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ON THE NUMERICAL SOLUTION OF SPHERICALLY SYMMETRIC PROBLEMS USING THE THEORY OF A COSSERAT SURFACE

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Abstract—A numerical solution of spherically symmetric problems is formulated by dividing the spherical region into N spherical shells and modelling each of these shells as a Cosserat surface. By coupling the motion of each of these shells with that of its neighbours we obtain a system of ordinary differential equations for functions of time only. Both the displacements and the contact stresses at the common boundaries are obtained. Specific constitutive equations are developed for linear motion of an isotropic elastic shell and the resulting mass and stiffness matrices are shown to be different from those obtained using the displacement approach and Galerkin methods. Comparison between Cosserat, Galerkin and exact solutions are shown for three examples which include both static and dynamic problems. In each of these examples, the Cosserat solution is superior to the Galerkin solution.

1. INTRODUCTION

The objective of this paper is to formulate a numerical solution procedure for analyzing spherical motion of a hollow or solid elastic sphere. Following notions developed in Ref. [1] for the numerical solution of one-dimensional continuum problems using the theory of a Cosserat point, we divide the spherical region into N connected spherical shells. Each of these shells is modelled by the theory of a Cosserat surface[2, 3] which provides the theoretical framework within which equations of motion and constitutive equations are derived.

A system of ordinary differential equations of time only are developed to describe the motion of the spherical region by coupling the motion of each shell with that of its neighbours and using appropriate boundary conditions. Kinematic coupling requires each shell to be in contact with its neighbours. Kinetic coupling (see eqn (13)) requires the force applied to the $(I-1)$th shell by the Ith shell to be equal in magnitude and opposite in direction to the force applied to the Ith shell by the $(I-1)$th shell. This kinetic coupling is similar to that used in direct finite element methods[4].

In the following sections, we briefly record the basic equations of the theory of a Cosserat surface appropriate for non-linear motion of a spherical shell. Then, the solution procedure is described and explicit expressions, eqns (14), are derived for the contact stresses applied to the inner and outer surfaces of each shell. In Section 4 the equations are linearized and specific constitutive eqns (20) are proposed.

Within the context of the Cosserat formulation, the constitutive coefficients for resultant forces and moments and the inertia coefficients must be specified by constitutive equations and are not necessarily obtained by integration of three-dimensional equations, such as in the Galerkin method described in the appendix. In Section 5, these constitutive coefficients are specified by comparing predictions of the Cosserat theory with exact results for the static problem of a pressurized hollow sphere and the dynamic problem of free vibration of a solid sphere. With these specifications, the resulting stiffness and mass matrices have a non-trivial dependence on the geometry of the shell and are different from those obtained by the Galerkin method. Finally, in Section 6, comparison between Cosserat, Galerkin and exact solutions are shown for three examples which include both static and dynamic problems. In each of these examples the Cosserat solution is superior to the Galerkin solution.

2. BASIC EQUATIONS

Consider a spherical region which is divided into N connected spherical shells. In its reference configuration, the Ith shell ($I = 1, 2, \ldots, N$) has an internal radius ξ_I and an external radius ξ_{I+1}. Here, we model each of these shells with the theory of a Cosserat surface. Details of this theory and its application to shells may be found in the article by Naghdi[2]. Within the context of this theory and with respect to the present configuration, at time t, the Ith shell is characterized by the position vector $\mathbf{r}_I$ of a material point on the Cosserat surface and by the director $\mathbf{d}_I$, which is usually identified with a material fibre through the thickness of the shell. A motion of the shell is defined by†

$$\mathbf{r}_I = \mathbf{r}_I(\theta^\alpha, t), \qquad \mathbf{d}_I = \mathbf{d}_I(\theta^\alpha, t) \tag{1a,b}$$

$$[\mathbf{a}_{1I} \quad \mathbf{a}_{2I} \quad \mathbf{d}_I] > 0 \tag{1c}$$

where θ^α ($\alpha = 1, 2$) are convected coordinates and $\mathbf{a}_{\alpha I}$ are tangent vectors to the surface. The vectors $\mathbf{a}_{\alpha I}$, together with the unit vector $\mathbf{a}_{3I}$ normal to the shell surface and the metric $a_I^{1/2}$ are defined by

$$\mathbf{a}_{\alpha I} = \frac{\partial \mathbf{r}_I}{\partial \theta^\alpha}, \qquad \mathbf{a}_{\alpha I} \cdot \mathbf{a}_{3I} = 0 \tag{2a, b}$$

$$\mathbf{a}_{3I} \cdot \mathbf{a}_{3I} = 1, \qquad [\mathbf{a}_{1I} \quad \mathbf{a}_{2I} \quad \mathbf{a}_{3I}] = a_I^{1/2} > 0. \tag{2c, d}$$

Further, the velocity $\mathbf{v}_I$ and director velocity $\mathbf{w}_I$ are given by

$$\mathbf{v}_I = \dot{\mathbf{r}}_I, \qquad \mathbf{w}_I = \dot{\mathbf{d}}_I \tag{3a, b}$$

where a superposed dot denotes material time differentiation holding θ^α fixed.

To describe spherical motion of a spherical shell, we introduce a spherical coordinate system with unit base vectors $(\mathbf{e}_\xi, \mathbf{e}_\theta, \mathbf{e}_\phi)$ defined by

$$\mathbf{e}_\xi = \sin\theta(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + \cos\theta\,\mathbf{e}_3 \tag{4a}$$

$$\mathbf{e}_\theta = \cos\theta(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) - \sin\theta\,\mathbf{e}_3 \tag{4b}$$

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2 \tag{4c}$$

relative to fixed Cartesian base vectors $\mathbf{e}_i$ ($i = 1, 2, 3$). Then, if we identify the convected coordinates θ^1, θ^2 with the angles θ, ϕ, respectively, we may write the position vector $\mathbf{r}_I$ and director $\mathbf{d}_I$ in the forms

$$\mathbf{r}_I = \bar{r}_I(t)\mathbf{e}_\xi(\theta, \phi), \qquad \mathbf{d}_I = d_I(t)\,\mathbf{e}_\xi(\theta, \phi) \tag{5a, b}$$

where $\bar{r}_I$, d_I are scalar functions of time only, with $\bar{r}_I$ representing the mean radius of the shell and d_I representing the shell thickness.

Using the notation in Ref. [3] and assuming that the assigned fields $\mathbf{f}_I$ and $\mathbf{l}$ have only components in the radial direction $(\mathbf{e}_\xi)$ which are independent of θ, ϕ, it can be shown that

† Throughout the text there is no sum on repeated capital indices which identify the Ith shell.

for the spherical motion, eqns (5), the non-linear balance laws of the theory of a Cosserat surface reduce to

$$\lambda_I = \rho_I a_I^{1/2} = \rho_{0I} A_I^{1/2} \tag{6a}$$

$$\rho_I(\ddot{\bar{r}}_I + y_I^1 \ddot{d}_I) = \rho_I f_I^3 - \frac{2N_{\theta\theta I}}{\bar{r}_I} \tag{6b}$$

$$\rho_I(y_I^1 \ddot{\bar{r}}_I + y_I^2 \ddot{d}_I) = \rho_I l_I^3 - k_I - \frac{2M_{\theta\theta I}}{\bar{r}_I} \tag{6c}$$

where eqn (6a) represents conservation of mass, and eqns (6b) and (6c) represent, respectively, balance of linear momentum and balance of director momentum. In eqns (6), ρ_I is the mass density (mass per unit surface area) in the present configuration; ρ_{0I} and $A_I^{1/2}$ are the reference values of ρ_I and $a_I^{1/2}$, respectively; $f_I^3 = \mathbf{f}_I \cdot \mathbf{a}_{3I}$; $l_I^3 = \mathbf{l}_I \cdot \mathbf{a}_{3I}$; y_I^1 and y_I^2 are inertia coefficients which are independent of time; $N_{\theta\theta I} = (\mathbf{a}_{1I} \cdot \mathbf{a}_{1I}) N_I^{11}$ and $M_{\theta\theta I} = (\mathbf{a}_{1I} \cdot \mathbf{a}_{1I}) M_I^{11}$ are the physical components of the resultant force N_I^{11} and resultant moment M_I^{11}, respectively; and $k_I = \mathbf{k}_I \cdot \mathbf{a}_{3I}$ is the intrinsic director couple.

For an elastic shell, the quantities $N_{\theta\theta I}$, k_I, $M_{\theta\theta I}$ are related to derivatives of a strain energy function ψ_I by the equations

$$\psi_I = \psi_I(e_{11I}, \gamma_{3I}, \kappa_{11I}) \tag{7a}$$

$$N_{\theta\theta I} = \rho_I \bar{r}_I^2 \frac{\partial \psi_I}{\partial e_{11I}} + \frac{d_I}{\bar{r}_I} M_{\theta\theta I} \tag{7b}$$

$$k_I = \rho_I \frac{\partial \psi_I}{\partial \gamma_{3I}}, \qquad M_{\theta\theta I} = \rho_I \bar{r}_I^2 \frac{\partial \psi_I}{\partial \kappa_{11I}} \tag{7c,d}$$

where the strain measures e_{11I}, γ_{3I}, κ_{11I} are defined by

$$e_{11I} = \frac{1}{2}(\bar{r}_I^2 - \bar{R}_I^2), \qquad \gamma_{3I} = d_I - D_I, \qquad \kappa_{11I} = \bar{r}_I d_I - \bar{R}_I D_I \tag{8a--c}$$

with $\bar{R}_I$, D_I being the reference values of $\bar{r}_I$, d_I, respectively, given by

$$\bar{R}_I = \frac{1}{2}(\xi_I + \xi_{I+1}), \qquad D_I = \xi_{I+1} - \xi_I. \tag{9a, b}$$

A one-to-one correspondence between the theory of a Cosserat surface and the three-dimensional theory can be established if we make the kinematic assumption that the position vector $\mathbf{r}^*$ locating material points in the Ith shell is given by

$$\mathbf{r}^* = \mathbf{r}_I + \theta_I \mathbf{d}_I \tag{10a}$$

$$\theta_I = \frac{2\xi - \xi_I - \xi_{I+1}}{2(\xi_{I+1} - \xi_I)}, \quad \text{for } \xi \in [\xi_I, \xi_{I+1}] \tag{10b}$$

where ξ is a convected coordinate in the radial direction, with $\theta_I = -1/2$ on the inner surface and $\theta_I = 1/2$ on the outer surface. It follows from eqns (5) and (10) that $\bar{r}_I$ and d_I are related to the inner radius r_I and outer radius r_{I+1} of the Ith shell by the equations

$$\bar{r}_I = \frac{1}{2}(r_I + r_{I+1}), \qquad d_I = r_{I+1} - r_I. \tag{11a, b}$$

Further, using the results in Ref. [2] it can be shown that the quantities ρ_{0I}, y_I^1, y_I^2, f_I^3, l_I^3 may be related to the three-dimensional mass density ρ_0^* in the reference configuration, the radial component $f^* = \mathbf{f}^* \cdot \mathbf{e}_\xi$ of the three-dimensional body force $\mathbf{f}^*$ (per unit mass), and the traction vectors: $\mathbf{t}_{1I} = t_{1I}\mathbf{e}_\xi$ on the inner surface and $\mathbf{t}_{2I} = t_{2I}\mathbf{e}_\xi$ on the outer surface by the equations

$$\rho_{0I}\bar{R}_I^2 = \int_{\xi_I}^{\xi_{I+1}} \rho_0^*(\bar{R}_I + \theta_I D_I)^2 \, \mathrm{d}\xi \tag{12a}$$

$$\rho_{0I}\bar{R}_I^2 y_I^1 = \int_{\xi_I}^{\xi_{I+1}} \rho_0^*(\bar{R}_I + \theta_I D_I)^2 \theta_I \, \mathrm{d}\xi \tag{12b}$$

$$\rho_{0I}\bar{R}_I^2 y_I^2 = \int_{\xi_I}^{\xi_{I+1}} \rho_0^*(\bar{R}_I + \theta_I D_I)^2 \theta_I^2 \, \mathrm{d}\xi \tag{12c}$$

$$\rho_{0I}\bar{R}_I^2 f_I^3 = \rho_{0I}\bar{R}_I^2 f_I + t_{1I} r_I^2 + t_{2I} r_{I+1}^2 \tag{12d}$$

$$\rho_{0I}\bar{R}_I^2 l_I^3 = \rho_{0I}\bar{R}_I^2 l_I - \frac{1}{2} t_{1I} r_I^2 + \frac{1}{2} t_{2I} r_{I+1}^2 \tag{12e}$$

$$\rho_{0I}\bar{R}_0^2 f_I = \int_{\xi_I}^{\xi_{I+1}} \rho_0^*(\bar{R}_I + \theta_I D_I)^2 f^* \, \mathrm{d}\xi \tag{12f}$$

$$\rho_{0I}\bar{R}_I^2 l_I = \int_{\xi_I}^{\xi_{I+1}} \rho_0^*(\bar{R}_I + \theta_I D_I)^2 f^* \theta_I \, \mathrm{d}\xi. \tag{12g}$$

In the following sections, we show that the solution of dynamic spherically symmetric problems can be significantly improved by specifying the inertia coefficients y_I^1 and y_I^2 by values different from those obtained by eqns (12b) and (12c).

3. SOLUTION PROCEDURE

Following the procedure proposed in Ref. [1] for the theory of a Cosserat point, we formulate the numerical solution of the dynamic response of the spherical region $(\xi_1 \leq \xi \leq \xi_{N+1})$ by coupling the response of the Ith shell to that of its neighbours. Specifically, we develop a system of ordinary differential equations of time only by using kinematic and kinetic coupling equations. Kinematic coupling is implied by eqns (11), with r_I being the outer radius of the $(I-1)$th shell and the inner radius of the Ith shell. Kinetic coupling requires the force applied to the $(I-1)$th shell by the Ith shell to be equal in magnitude and opposite in direction to the force applied to the Ith shell by the $(I-1)$th shell. This requires

$$t_{2I-1} + t_{1I} = 0 \quad \text{for } I = 2, 3, \ldots, N. \tag{13}$$

Once constitutive equations are given for ρ_{0I}, y_I^1, y_I^2, ψ_I, and the assigned fields f_I, l_I and the contact stresses t_{1I}, t_{2I} are specified, eqns (6b) and (6c) represent two equations to determine the two unknown radii r_I, r_{I+1} (use of eqns (6a) and (11) is implied).

Alternatively, for arbitrary values of these quantities eqns (6b) and (6c) may be solved for t_{1I} and t_{2I} to obtain

$$r_I^2 t_{1I} = -\rho_{0I}\bar{R}_I^2\left(\frac{1}{2}f_I - l_I\right) + \bar{R}_I^2\left[\frac{1}{2}\left(\frac{2N_{\theta\theta I}}{\bar{r}_I}\right) - \left(k_I + \frac{2M_{\theta\theta I}}{\bar{r}_I}\right)\right]$$
$$+ \rho_{0I}\bar{R}_I^2\left[\left(\frac{1}{4} - y_I^1 + y_I^2\right)\ddot{r}_I + \left(\frac{1}{4} - y_I^2\right)\ddot{r}_{I+1}\right] \tag{14a}$$

$$r_{I+1}^2 t_{2I} = -\rho_{0I}\bar{R}_I^2\left(\frac{1}{2}f_I + l_I\right) + \bar{R}_I^2\left[\frac{1}{2}\left(\frac{2N_{\theta\theta I}}{\bar{r}_I}\right) + \left(k_I + \frac{2M_{\theta\theta I}}{\bar{r}_I}\right)\right]$$
$$+ \rho_{0I}\bar{R}_I^2\left[\left(\frac{1}{4} - y_I^2\right)\ddot{r}_I + \left(\frac{1}{4} + y_I^1 + y_I^2\right)\ddot{r}_{I+1}\right] \tag{14b}$$

where we have used eqns (2), (5), (11), (12d) and (12e) and have written conservation of mass, eqn (6a), in the form

$$\rho_I \bar{r}_I^2 = \rho_{0I}\bar{R}_I^2. \tag{15}$$

Substitution of eqns (14) into eqn (13) yields $(N-1)$ coupling equations to determine the $(N+1)$ unknowns r_I. The remaining two equations for r_I are obtained by specifying boundary conditions at the inner ($\xi = \xi_1$) and outer ($\xi = \xi_{N+1}$) surfaces of the spherical region. These boundary conditions require specification of either position or contact stress at the inner and outer boundaries. Thus, we specify

$$[r_1 = \hat{r}_1(t) \quad \text{or} \quad t_{11} = \hat{t}_{11}(t)] \tag{16a}$$

and

$$[r_{N+1} = \hat{r}_{N+1}(t) \quad \text{or} \quad t_{2N} = \hat{t}_{2N}(t)] \tag{16b}$$

where $\hat{r}_1$, $\hat{t}_{11}$, $\hat{r}_{N+1}$, $\hat{t}_{2N}$ are specified functions of time only. If, say, the position r_1 is specified, then eqn (14a) for $I = 1$ determines the value for t_{11}. On the other hand, if, say, the contact stress t_{11} is specified, then eqn (14a) for $I = 1$ is a differential equation for r_1. Once initial conditions for r_I, $\dot{r}_I$, and boundary conditions (16) are specified, the equations resulting from eqn (13), eqn (17a) for $I = 1$ and eqn (17b) for $I = N$, determine the $(N+1)$ unknowns (r_1 or t_{11}), (r_{N+1} or t_{2N}) and r_I ($I = 2, 3, \ldots, N$). Then the remaining contact stresses t_{1I} and t_{2I} may be determined by eqns (14).

4. LINEAR THEORY

For linear theory it is convenient to introduce the displacement U_I from the reference position ξ_I such that

$$r_I = \xi_I + U_I. \tag{17}$$

Assuming that the displacement U_I is small, the linearized form of the equations of motion, eqns (14), may be obtained by merely replacing r_I, $\bar{r}_I$, $\ddot{r}_I$ by ξ_I, $\bar{R}_I$, $\ddot{U}_I$, respectively. Further, neglecting quadratic terms in the displacement U_I and using eqns (9), (11) and (17) the linear strains, eqns (8), become

$$e_{11I} = \frac{1}{2}\bar{R}_I(U_I + U_{I+1}), \qquad \gamma_{3I} = U_{I+1} - U_I \tag{18a,b}$$

$$\kappa_{11I} = \bar{R}_I(U_{I+1} - U_I) + \frac{1}{2}D_I(U_I + U_{I+1}). \tag{18c}$$

Linear constitutive equations for an isotropic, elastic shell with symmetry about the reference surface have been discussed in detail by Naghdi[2]. Here, we merely recall that a strain energy function ψ_I can be specified such that

$$N_{\theta\theta I} = \frac{2(\alpha_{1I} + \alpha_{2I})}{\bar{R}_I^2} e_{11I} + \alpha_{9I}\gamma_{3I} + \frac{D_I}{\bar{R}_I} M_{\theta\theta I} \tag{19a}$$

$$k_I = \alpha_{4I}\gamma_{3I} + \frac{2\alpha_{9I}}{\bar{R}_I^2} e_{11I} \tag{19b}$$

$$M_{\theta\theta I} = \frac{(2\alpha_{5I} + \alpha_{6I} + \alpha_{7I})}{\bar{R}_I^2} \kappa_{11I} \tag{19c}$$

where the functions $\alpha_I = \{\alpha_{1I}, \alpha_{2I}, \alpha_{4I} - \alpha_{7I}, \alpha_{9I}\}$ characterize the response of the Ith shell and may depend on the reference geometry of the shell through the quantities $\bar{R}_I$ and D_I. It follows, using eqns (18) and (19) that

$$\frac{2N_{\theta\theta I}}{\bar{R}_I} = C_{1I}U_I + C_{2I}U_{I+1} \tag{20a}$$

$$k_I + \frac{2M_{\theta\theta I}}{\bar{R}_I} = C_{3I}U_I + C_{4I}U_{I+1} \tag{20b}$$

where $C_{1I} - C_{4I}$ are functions of the quantities α_I, $\bar{R}_I$, and D_I. Since the equations of motion, eqns (14), depend on $N_{\theta\theta I}$, k_I and $M_{\theta\theta I}$ only through expressions of the form of eqns (20), the relevant response of the shell is specified by the constitutive coefficients C_{1I}–C_{4I}. Consequently, there is no need to determine each of the quantities α_I.

With the help of eqns (20) the linearized forms of eqns (14) become

$$t_{1I} = -\rho_{0I}\left(\frac{1}{2}f_I - l_I\right) + \left(\frac{1}{2}C_{1I} - C_{3I}\right)U_I + \left(\frac{1}{2}C_{2I} - C_{4I}\right)U_{I+1} + \rho_{0I}\left[\left(\frac{1}{4} - y_I^1 + y_I^2\right)\ddot{U}_I + \left(\frac{1}{4} - y_I^2\right)\ddot{U}_{I+1}\right] \tag{21a}$$

$$t_{2I} = -\rho_{0I}\left(\frac{1}{2}f_I + l_I\right) + \left(\frac{1}{2}C_{1I} + C_{3I}\right)U_I + \left(\frac{1}{2}C_{2I} + C_{4I}\right)U_{I+1} + \rho_{0I}\left[\left(\frac{1}{4} - y_I^2\right)\ddot{U}_I + \left(\frac{1}{4} + y_I^1 + y_I^2\right)\ddot{U}_{I+1}\right]. \tag{21b}$$

Then, substituting eqns (21) into the kinetic coupling eqn (13) and using eqn (21a) for $I = 1$ and eqn (21b) for $I = N$, the equations of motion of the linear theory may be reduced to

$$\sum_{J=1}^{N+1} (M_{IJ}\ddot{U}_J + K_{IJ}U_J) = b_I, \qquad I = 1, 2, \ldots, N+1 \tag{22}$$

where M_{IJ} is a symmetric, positive definite mass matrix, K_{IJ} is a stiffness matrix,† and b_I is a load vector, each given by

$$M_{11} = \rho_{01}\bar{R}_1^2\left(\frac{1}{4} - y_1^1 + y_1^2\right) \tag{23a}$$

$$M_{N+1,N+1} = \rho_{0N}\bar{R}_N^2\left(\frac{1}{4} + y_N^1 + y_N^2\right) \tag{23b}$$

$$M_{II} = \rho_{0I-1}\bar{R}_{I-1}^2\left(\frac{1}{4} + y_{I-1}^1 + y_{I-1}^2\right) + \rho_{0I}\bar{R}_I^2\left(\frac{1}{4} - y_I^1 + y_I^2\right),$$
$$\text{for } I = 2, 3, \ldots, N \tag{23c}$$

$$M_{I,I+1} = M_{I+1,I} = \rho_{0I}\bar{R}_I^2\left(\frac{1}{4} - y_I^2\right), \qquad \text{for } I = 1, 2, \ldots, N \tag{23d}$$

$$\text{all other } M_{IJ} = 0 \tag{23e}$$

$$K_{11} = \bar{R}_1^2\left(\frac{1}{2}C_{11} - C_{31}\right) \tag{24a}$$

$$K_{N+1,N+1} = \bar{R}_N^2\left(\frac{1}{2}C_{2N} + C_{4N}\right) \tag{24b}$$

$$K_{II} = \bar{R}_{I-1}^2\left(\frac{1}{2}C_{2I-1} + C_{4I-1}\right) + \bar{R}_I^2\left(\frac{1}{2}C_{1I} - C_{3I}\right), \qquad \text{for } I = 2, 3, \ldots, N \tag{24c}$$

$$K_{I,I+1} = \bar{R}_I^2\left(\frac{1}{2}C_{2I} - C_{4I}\right) \tag{24d}$$

$$K_{I+1,I} = \bar{R}_I^2\left(\frac{1}{2}C_{1I} + C_{3I}\right), \qquad \text{for } I = 1, 2, \ldots, N \tag{24e}$$

$$\text{all other } K_{IJ} = 0 \tag{24f}$$

† It will be shown in the next section that K_{IJ} is symmetric.

$$b_1 = \rho_{01}\bar{R}_1^2\left(\frac{1}{2}f_1 - l_1\right) + \xi_1^2 t_{11} \tag{25a}$$

$$b_{N+1} = \rho_{0N}\bar{R}_N^2\left(\frac{1}{2}f_N + l_N\right) + \xi_{N+1}^2 t_{2N} \tag{25b}$$

$$b_I = \rho_{0I-1}\bar{R}_{I-1}^2\left(\frac{1}{2}f_{I-1} + l_{I-1}\right) + \rho_{0I}\bar{R}_I^2\left(\frac{1}{2}f_I - l_I\right), \qquad \text{for } I = 2, 3, \ldots, N. \tag{25c}$$

In the following section, we determine explicit constitutive equations for C_{1I}–C_{4I}, ρ_{0I}, y_I^1, y_I^2.

Once boundary conditions (16) and initial conditions for U_I, $\dot{U}_I$ are specified, eqns (22) represent $(N + 1)$ equations to determine the $(N + 1)$ unknowns (U_1 or t_{11}), (U_{N+1} or t_{2N}) and U_I ($I = 2, 3, \ldots, N$). Then the remaining contact stresses t_{1I} and t_{2I} can be determined by eqns (21).

5. DETERMINATION OF THE CONSTITUTIVE COEFFICIENTS

In this section we determine expressions for the constitutive coefficients C_{1I}–C_{4I}, ρ_{0I}, y_I^1, y_I^2 by comparing Cosserat predictions for a single shell with available static and dynamic exact solutions. First, the stiffness coefficients C_{1I}–C_{4I} are determined by comparing with a static solution and then the inertia coefficients ρ_{0I}, y_I^1, y_I^2 are determined by comparing with a dynamic solution.

5.1. *Static solution*

Consider the static problem of a hollow spherical shell with inner radius ξ_I and outer radius ξ_{I+1} which is subjected to an internal pressure P_{1I} and an external pressure P_{2I}. Confining attention to the Ith shell, neglecting body forces ($f_I = 0$, $l_I = 0$), and using expressions (20), the static forms of eqns (21) reduce to

$$\left(\frac{1}{2}C_{1I} - C_{3I}\right)U_I + \left(\frac{1}{2}C_{2I} - C_{4I}\right)U_{I+1} = t_{1I} = P_{1I} \tag{26a}$$

$$\left(\frac{1}{2}C_{1I} + C_{3I}\right)U_I + \left(\frac{1}{2}C_{2I} + C_{4I}\right)U_{I+1} = t_{2I} = -P_{2I}. \tag{26b}$$

Now, coefficients C_{1I}–C_{4I} can be determined by comparing eqns (26) with similar expressions derived from the exact solution (Ref. [5], Section 94). Using these results, together with expressions (23) we deduce that

$$\frac{1}{2}C_{1I} - C_{3I} = \left(\frac{3\lambda + 2\mu}{\bar{R}_I^2}\right)\left(\frac{\xi_I^4}{\xi_{I+1}^3 - \xi_I^3}\right) + \left(\frac{4\mu}{\bar{R}_I^2}\right)\left(\frac{\xi_I\xi_{I+1}^3}{\xi_{I+1}^3 - \xi_I^3}\right) \tag{27a}$$

$$\frac{1}{2}C_{1I} + C_{3I} = -\frac{3(\lambda + 2\mu)}{\bar{R}_I^2}\left(\frac{\xi_I^2\xi_{I+1}^2}{\xi_{I+1}^3 - \xi_I^3}\right) \tag{27b}$$

$$\frac{1}{2}C_{2I} - C_{4I} = \frac{1}{2}C_{1I} + C_{3I} \tag{27c}$$

$$\frac{1}{2}C_{2I} + C_{4I} = \left(\frac{3\lambda + 2\mu}{\bar{R}_I^2}\right)\left(\frac{\xi_{I+1}^4}{\xi_{I+1}^3 - \xi_I^3}\right) + \left(\frac{4\mu}{\bar{R}_I^2}\right)\left(\frac{\xi_I^3\xi_{I+1}}{\xi_{I+1}^3 - \xi_I^3}\right) \tag{27d}$$

and

$$K_{11} = (3\lambda + 2\mu)\left(\frac{\xi_1^4}{\xi_2^3 - \xi_1^3}\right) + 4\mu\left(\frac{\xi_1\xi_2^3}{\xi_2^3 - \xi_1^3}\right) \tag{28a}$$

$$K_{N+1,N+1} = (3\lambda + 2\mu)\left(\frac{\xi_{N+1}^4}{\xi_{N+1}^3 - \xi_N^3}\right) + 4\mu\left(\frac{\xi_N^3\xi_{N+1}}{\xi_{N+1}^3 - \xi_N^3}\right) \tag{28b}$$

$$K_{II} = (3\lambda + 2\mu)\left(\frac{\xi_I^4}{\xi_I^3 - \xi_{I-1}^3}\right) + 4\mu\left(\frac{\xi_{I-1}^3\xi_I}{\xi_I^3 - \xi_{I-1}^3}\right)$$
$$+ (3\lambda + 2\mu)\left(\frac{\xi_I^4}{\xi_{I+1}^3 - \xi_I^3}\right) + 4\mu\left(\frac{\xi_I\xi_{I+1}^3}{\xi_{I+1}^3 - \xi_I^3}\right), \qquad \text{for } I = 2, 3, \ldots, N \tag{28c}$$

$$K_{I,I+1} = K_{I+1,I} = -3(\lambda + 2\mu)\left(\frac{\xi_I^2\xi_{I+1}^2}{\xi_{I+1}^3 - \xi_I^3}\right), \qquad \text{for } I = 1, 2, \ldots, N \tag{28d}$$

$$\text{all other } K_{IJ} = 0 \tag{28e}$$

where λ, μ are Lamé's constants for the isotropic elastic solid. Here, we have decided to match values of the inner radius r_I (associated with U_I) and outer radius r_{I+1} (associated with U_{I+1}) of the Ith shell, instead of matching values of the mean radius $\bar{r}_I$ and the thickness d_I, because r_I, r_{I+1} determine the locations of the contact surfaces between the Ith shell and its neighbours. It is important to note that expressions (28) are different from those obtained in eqns (A10) using Galerkin methods which integrate the three-dimensional constitutive equation.

5.2. *Dynamic solution*

Consider the problem of free vibration of a solid sphere of radius D_1 and constant mass density ρ_0^* (mass per unit volume). Confining attention to a single shell ($N = 1$), neglecting body forces ($f_1 = 0$, $l_1 = 0$), and setting $U_1 = 0$, eqns (22) reduce to

$$M_{12}\ddot{U}_2 = 0, \qquad M_{22}\ddot{U}_2 + K_{22}U_2 = 0 \tag{29a,b}$$

where M_{12}, M_{22} are determined by eqns (23) and K_{22} is determined by eqns (28). Obviously, to obtain a non-trivial solution we must require M_{12} to vanish. In this regard we note that, within the context of the Cosserat theory, the inertia quantities ρ_{0I}, y_I^1, y_I^2 need not be determined by expressions (12a)–(12c). However, expressions (12a)–(12c) provide guidance for the specification of ρ_{0I}, y_I^1, y_I^2. Specifically, let us assume that ρ_{0I} predicts the correct mass density (mass per unit area) of the Ith shell and that the form of the dependence of y_I^1 and y_I^2 on $\bar{R}_I$, D_I is given by expressions (12b) and (12c), but the coefficients are to be determined by constitutive equations. Thus, using eqns (10b) in expressions (12a)–(12c) we obtain

$$\rho_{0I} = \rho_0^* D_I\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) \tag{30a}$$

$$\rho_{0I}y_I^1 = a\rho_0^*\frac{D_I^2}{\bar{R}_I} \tag{30b}$$

$$\rho_{0I}y_I^2 = b\rho_0^* D_I\left(1 + c\frac{D_I^2}{\bar{R}_I^2}\right) \tag{30c}$$

where a, b, c are constants independent of $\bar{R}_I$, D_I, which must be specified. The choice of the constants a, b, c is not totally arbitrary. In particular, we require the expression for the specific (per unit mass) kinetic energy κ_I of the Ith Cosserat surface

$$\kappa_I = \frac{1}{2}(\mathbf{v}_I \cdot \mathbf{v}_I + 2y_I^1 \mathbf{v}_I \cdot \mathbf{w}_I + y_I^2 \mathbf{w}_I \cdot \mathbf{w}_I] \tag{31}$$

to be positive definite. Rewriting eqn (31) in the form

$$\kappa_I = \frac{1}{2}[(\mathbf{v}_I + y_I^1 \mathbf{w}_I) \cdot (\mathbf{v}_I + y_I^1 \mathbf{w}_I) + \{y_I^2 - (y_I^1)^2\} \mathbf{w}_I \cdot \mathbf{w}_I] \tag{32}$$

we realize that a sufficient condition for positive definiteness of κ_I is

$$y_I^2 - (y_I^1)^2 > 0 \tag{33}$$

which places restrictions on a, b, c.

The value of the constant b is specified by considering the limit of a very thin shell ($D_I/\bar{R}_I \to 0$) and requiring the limiting value of y_I^2 to be the same as the value $y^{11} = 1/\pi^2$ suggested in Ref. [6] in the analysis of vibration of a parallelpiped using the theory of a Cosserat point. Thus, we take

$$b = \frac{1}{\pi^2}. \tag{34}$$

The value of the constant c may be specified by requiring M_{12} to vanish for a single ($N = 1$) solid ($D_1/\bar{R}_1 = 2$) shell. Thus, using eqns (23d), (30a), (30c) and (34) we obtain

$$c = \frac{\pi^2 - 3}{12}. \tag{35}$$

The value of a is specified by comparing the natural frequency ω predicted by the Cosserat eqn (29b), namely

$$\omega^2 = \frac{K_{22}}{M_{22}} = \frac{6(3\lambda + 2\mu)}{\rho_0^* D_1^2 (1 + 3a)} \tag{36}$$

with the lowest frequency ω_1^* predicted by the exact solution (Ref. [7], Section 196). For the general case (λ not necessarily equal to μ) the exact solution yields

$$(\omega_m^*) = \left(\frac{\lambda + 2\mu}{\rho_0^*}\right)\left(\frac{B_m^*}{D_1}\right)^2, \qquad \frac{\tan B_m^*}{B_m^*} = \frac{1}{1 - \left(\frac{\lambda + 2\mu}{4\mu}\right)(B_m^*)^2} \tag{37a,b}$$

where ω_m^* is the frequency associated with the mth non-zero root B_m^* of eqn (37b). It follows from eqns (36) and (37) that

$$\left(\frac{\omega}{\omega_1^*}\right)^2 = \left(\frac{6}{1 + 3a}\right)\left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\left(\frac{1}{B_1^*}\right)^2. \tag{38}$$

Physically, we cannot require ω to equal ω_1^* for all values of λ, μ because the inertia quantity a would have to be a function of the material constants λ, μ. However, the value of a obtained by solving eqn (38) with $\omega = \omega_1^*$ is reasonably constant ($a \in [0.174\text{–}0.223]$)

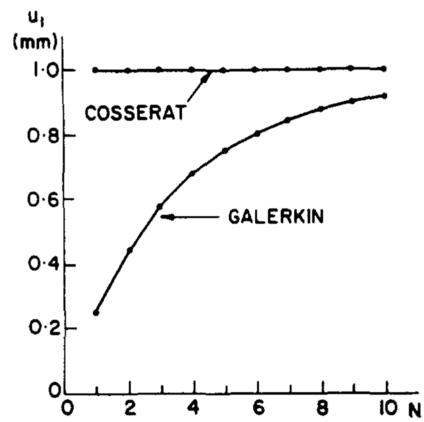

Fig. 1. Static solution for the displacement U_1 of the inner surface of an internally pressurized hollow sphere which is modelled by N spherical shells.

for values of Poisson's ratio $\nu = \lambda/2(\lambda + \mu)$ between 0.25 and 0.4. Consequently, we suggest the constant value

$$a = \frac{1}{5} \tag{39}$$

to be an average value for this range of material properties. Now, using specifications (34), (35), (39) the inertia quantities (30) and the mass matrix (23) are determined, and condition (33) is satisfied for all possible values of $D_I/\bar{R}_I$ (this quantity is between 0 and 2).

Finally, we note that integrals (12a)–(12c) yield the expressions (30) with

$$a = \frac{1}{6}, \qquad b = \frac{1}{12}, \qquad c = \frac{3}{20}. \tag{40a–c}$$

These values are the same as those obtained for the Galerkin solution of the appendix. They also yield a non-zero value for M_{12} in eqn (29a), which means that a constraint force must be introduced at the centre of the solid sphere to obtain non-trivial solutions.

6. EXAMPLES

In this section, we consider three examples to compare the results predicted by the Cosserat formulation [eqn (22) with specifications (28), (34), (35) and (39)] with the Galerkin formulation of the appendix. A piecewise linear interpolation function is used in the Galerkin formulation to be consistent with the kinematics (10) of the Cosserat theory. For each of these examples we specify representative material properties by

$$\rho_0^* = 5000\,\mathrm{kg\,m^{-3}}, \qquad \lambda = \mu = 50\,\mathrm{GPa} \tag{41a,b}$$

and divide the spherical region into N shells of equal thickness.

6.1. *Example 1*

Consider the static problem of a hollow shell with inner radius 10 mm and outer radius 60 mm subjected to an internal pressure $P_1 = 20\,\mathrm{GPa}$ and zero external pressure. This high value of internal pressure was chosen merely for the convenience of predicting an inner displacement of 1 mm. Figure 1 shows the displacement U_1 of the inner surface predicted by the Cosserat and the Galerkin solutions when the shell is divided into $N = 1, 2, \ldots, 10$ layers. Since the constitutive coefficients C_{1I}–C_{4I} were specified by

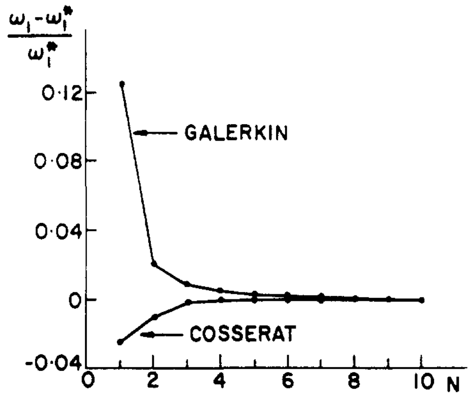

Fig. 2. Comparison between the first frequency ω_1 predicted by the Cosserat and Galerkin solutions and the exact frequency ω_1^* of free vibration of a solid sphere which is modelled by N spherical shells.

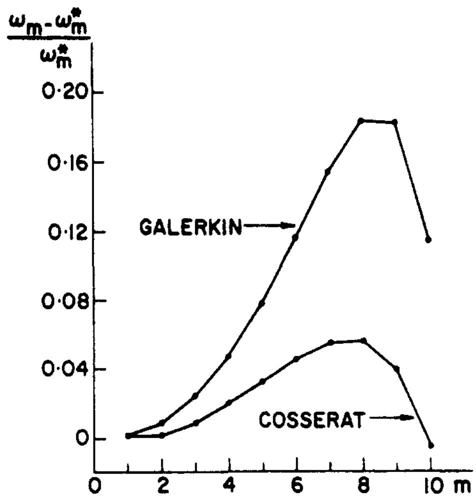

Fig. 3. Comparison between the mth frequency ω_m predicted by the Cosserat and Galerkin solutions and the exact frequency ω_m^* of free vibration of a solid sphere which is modelled by $N = 10$ spherical shells.

comparing with an exact solution of this type, the predictions of the Cosserat solution are exact for any number of layers N. In contrast, we observe that the predictions of the Galerkin solution have considerable error in the thick-shell range (low values of N).

6.2. *Example 2*

Consider the dynamic problem of free vibration of a solid sphere. Since both the Cosserat and Galerkin solutions predict the correct dependence on the radius of the sphere no value is specified for this radius. Figure 2 shows the error in the Cosserat and Galerkin predictions of the first frequency ω_1 relative to the exact frequency ω_1^* when the sphere is divided into $N = 1, 2, \ldots, 10$ layers. Both the Cosserat and the Galerkin solutions converge relatively rapidly to the exact solution. We observe that for a single layer ($N = 1$) the Cosserat prediction is not exact because the value of a in eqn (39) is not associated with the specification of material properties ($\nu = 0.25$, $\lambda = \mu$). We also investigated the relative error in the prediction of the mth natural frequency ω_m when the shell was divided into $N = 10$ layers and the results are shown in Fig. 3. The Cosserat predictions are within 6% error whereas those of the Galerkin solution are as high as 18% in error.

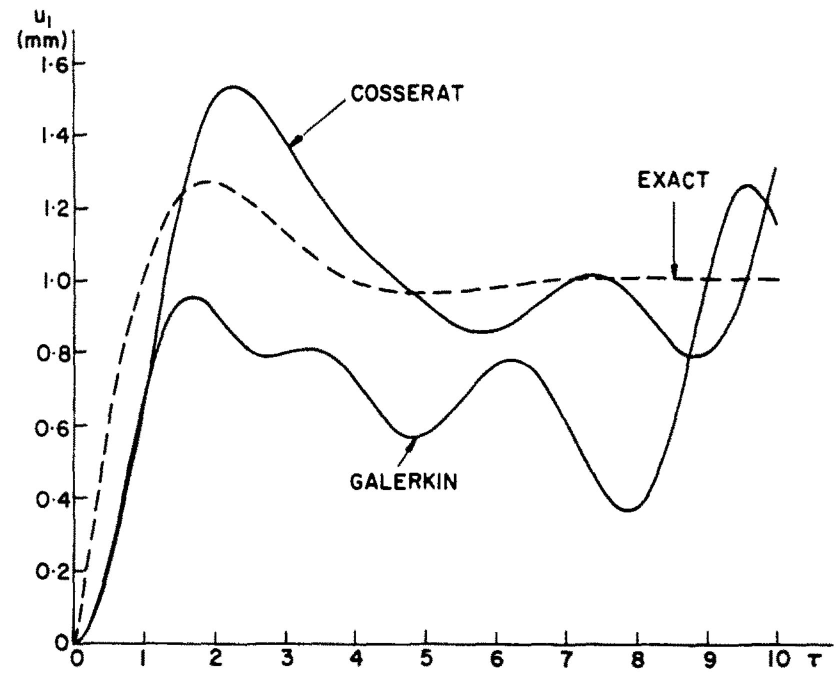

Fig. 4. Dynamic solution for the displacement U_1 of the inner surface of an internally pressurized hollow sphere which is modelled by $N = 4$ spherical shells. The parameter τ is a non-dimensional time defined by eqn (42a).

6.3. *Example 3*

In a sense, it is unfair to use the previous two examples for comparison purposes because these examples were used to obtain values for the constitutive coefficients of the Cosserat theory.† Consequently, here, we focus attention on a different dynamic problem and consider the hollow sphere of example 1, but now assume that it is initially at rest and is undeformed when the constant internal pressure P_1 is applied. Both the Cosserat and Galerkin solutions were calculated and compared with the exact solution for an infinite media with a spherical cavity reported in Ref. [8], p. 298.‡ The displacement U_1 of the inner surface is plotted in Figs 4 and 5 for the cases when the shell is divided into $N = 4$ and 10 layers, respectively. The non-dimensional time parameter τ is defined by

$$\tau = \frac{C_1 t}{\xi_1}, \qquad C_1^2 = \frac{\lambda + 2\mu}{\rho_0^*} \tag{42a,b}$$

where $\xi_1 = 10\,\text{mm}$ is the internal radius of the spherical cavity and C_1 is the dilatational wave speed. Since reflections from the free surface should not influence the displacement U_1 at the inner surface until the wave has traveled twice the thickness ($5\xi_1$) of the shell we have shown comparisons up to $\tau = 10$.

From Figs 4 and 5 we observe that the exact solution approaches the static solution by about $\tau = 7$. For the rather crude mesh ($N = 4$), both the Cosserat and the Galerkin solutions predict a slower initial response than the exact solution and the maximum value of the displacement is overpredicted by the Cosserat solution and underpredicted by the Galerkin solution. For the more refined mesh ($N = 10$), the Cosserat solution follows the exact solution fairly closely until about $\tau = 7$ and the Galerkin solution again underpredicts

† For example 2 we only used the prediction of the first frequency of vibration to determine the constitutive coefficient a in eqn (39). Therefore, comparison between predictions of the higher frequencies is meaningful.

‡ The minus sign preceding the exponential term in eqn (5.3.38) of Ref. [8] should be changed to a plus sign.

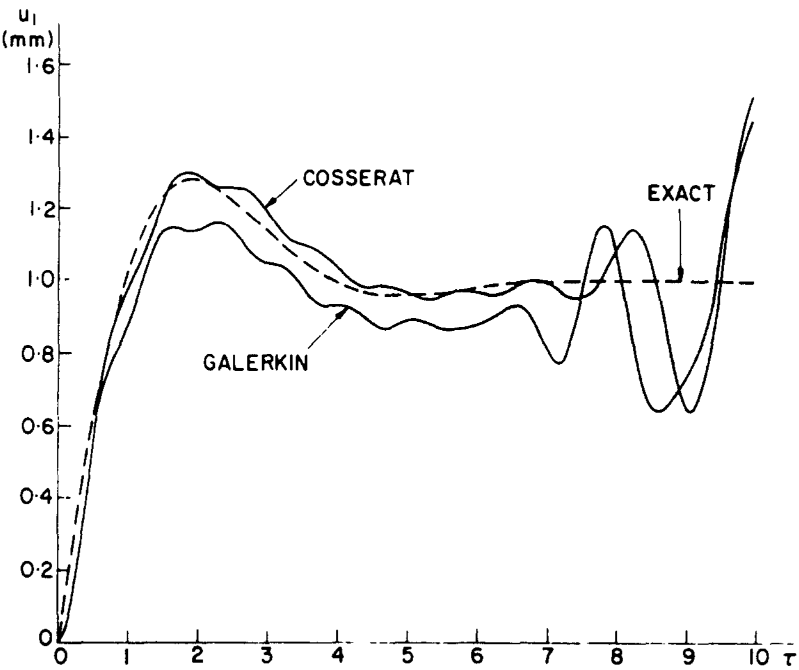

Fig. 5. Dynamic solution for the displacement U_1 of the inner surface of an internally pressurized hollow sphere which is modelled by $N = 10$ spherical shells. The parameter τ is a non-dimensional time defined by eqn (42a).

the maximum displacement. It is also of interest to note that the reflection from the free surface is observed earlier in the Galerkin solution than in the Cosserat solution.

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APPENDIX: GALERKIN SOLUTION

For spherically symmetric motion of an isotropic material the only non-trivial equation of motion is

$$\rho_0^* \frac{\partial^2 U^*}{\partial t^2} = \rho_0^* f^* + \frac{\partial t_{\xi\xi}}{\partial \xi} + \frac{2t_{\xi\xi} - t_{\phi\phi} - t_{\theta\theta}}{\xi} \tag{A1}$$

where U^* and f^* are the displacement and body force, respectively, in the positive $\mathbf{e}_\xi$ direction and $t_{\xi\xi}$, $t_{\phi\phi}$, $t_{\theta\theta}$ are stresses referred to the base vectors (4). Substituting the constitutive equations

$$t_{\xi\xi} = (\lambda + 2\mu)\frac{\partial U^*}{\partial \xi} + 2\lambda \frac{U^*}{\xi} \tag{A2a}$$

$$t_{\phi\phi} = t_{\theta\theta} = \lambda \frac{\partial U^*}{\partial \xi} + 2(\lambda + \mu)\frac{U^*}{\xi} \tag{A2b}$$

into eqn (A1) we obtain the displacement equation

$$\rho_0^* \frac{\partial U^*}{\partial t^2} = \rho_0^* f^* + \left(\frac{\lambda + 2\mu}{\xi^2}\right)\left[\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial U^*}{\partial \xi}\right) - 2U^*\right]. \tag{A3}$$

To develop a Galerkin solution, we divide the region of space into N parts whose endpoints are defined by ξ_I ($I = 1, 2, \ldots, N + 1$) and introduce the weighting functions $\phi_I(\xi)$ by

$$\phi_1(\xi) = \left(\frac{1}{2} - \theta_1\right) \quad \text{for } \xi \in [\xi_1, \xi_2] \tag{A4a}$$

$$\phi_I(\xi) = \begin{cases} \left(\theta_{I-1} + \dfrac{1}{2}\right) & \text{for } \xi \in [\xi_{I-1}, \xi_I] \\ \left(\dfrac{1}{2} - \theta_I\right) & \text{for } \xi \in [\xi_I, \xi_{I+1}] \end{cases} \tag{A4b}$$

$$\phi_{N+1}(\xi) = \left(\theta_N + \frac{1}{2}\right) \quad \text{for } \xi \in [\xi_N, \xi_{N+1}] \tag{A4c}$$

where the weighting functions are zero outside of these regions and where θ_I is defined by eqn (10b). Now, multiplying eqn (A3) by $\xi^2\phi_I$ and integrating over the region $[\xi_1, \xi_{N+1}]$ we have

$$\int_{\xi_1}^{\xi_{N+1}} \rho_0^* \frac{\partial U^*}{\partial \xi} \phi_I \xi^2 \, d\xi - \int_{\xi_1}^{\xi_{N+1}} \rho_0^* f^* \phi_I \xi^2 \, d\xi + (\lambda + 2\mu)\int_{\xi_1}^{\xi_{N+1}} \left(\xi^2 \frac{\partial U^*}{\partial \xi}\frac{d\phi_I}{d\xi} + 2U^*\phi_I\right) d\xi - (\lambda + 2\mu)\left[\xi^2 \frac{\partial U^*}{\partial \xi}\phi_I\right]_{\xi=\xi_1}^{\xi=\xi_{N+1}} = 0. \tag{A5}$$

Substituting the constitutive eqn (A2a) into the last term in eqn (A5) and assuming that the displacement U^* admits the representation

$$U^* = \sum_{J=1}^{N+1} U_J(t)\phi_J(\xi) \tag{A6}$$

where $U_J(t)$ is the displacement at ξ_J, we may write the equations of motion (A5) in the form of eqn (22) where

$$M_{IJ} = \int_{\xi_1}^{\xi_{N+1}} \rho_0^* \phi_I \phi_J \xi^2 \, d\xi \tag{A7a}$$

$$K_{IJ} = (\lambda + 2\mu)\int_{\xi_1}^{\xi_{N+1}} \left(\xi^2 \frac{d\phi_I}{d\xi}\frac{d\phi_J}{d\xi} + 2\phi_I\phi_J\right) d\xi + 2\lambda[-\xi_1\delta_{1I}\delta_{1J} + \xi_{N+1}\delta_{I,N+1}\delta_{J,N+1}] \tag{A7b}$$

$$b_I = \int_{\xi_1}^{\xi_{N+1}} \rho_0^* f^* \phi_I \xi^2 \, d\xi + [\xi_1^2 t_{11}\delta_{1I} + \xi_{N+1}^2 t_{2N}\delta_{I,N+1}]. \tag{A7c}$$

In eqn (A7) δ_{IJ} is the Kronecker delta symbol and we have specified the boundary values of $t_{\xi\xi}$ by

$$t_{\xi\xi}|_{\xi=\xi_1} = -t_{11}, \qquad t_{\xi\xi}|_{\xi=\xi_{N+1}} = t_{2N}. \tag{A8a,b}$$

It follows from eqns (9) and (10b) that

$$\xi = \bar{R}_I\left(1 + \theta_I \frac{D_I}{\bar{R}_I}\right) \quad \text{for } \xi \in [\xi_I, \xi_{I+1}]. \tag{A9}$$

Consequently, M_{IJ} takes the form of eqns (23) with a, b, c specified by eqns (40), b_I takes the form of eqns (25) with f_I, l_I defined by eqns (12f) and (12g) and K_{IJ} takes the form

$$K_{11} = (\lambda + 2\mu)\left(\frac{\bar{R}_1^2}{D_1}\right)\left[\left(1 + \frac{1}{12}\frac{D_1^2}{\bar{R}_1^2}\right) + \frac{2}{3}\left(\frac{D_1}{\bar{R}_1}\right)^2\right] - 2\lambda\bar{R}_1\left(1 - \frac{1}{2}\frac{D_1}{\bar{R}_1}\right) \tag{A10a}$$

$$K_{N+1,N+1} = (\lambda + 2\mu)\left(\frac{\bar{R}_N^2}{D_N}\right)\left[\left(1 + \frac{1}{12}\frac{D_N^2}{\bar{R}_N^2}\right) + \frac{2}{3}\left(\frac{D_N}{\bar{R}_N}\right)^2\right] + 2\lambda\bar{R}_N\left(1 + \frac{1}{2}\frac{D_N}{\bar{R}_N}\right) \tag{A10b}$$

$$K_{II} = (\lambda + 2\mu)\left(\frac{\bar{R}_{I-1}^2}{D_{I-1}}\right)\left[\left(1 + \frac{1}{12}\frac{D_{I-1}^2}{\bar{R}_{I-1}^2}\right) + \frac{2}{3}\left(\frac{D_{I-1}}{\bar{R}_{I-1}}\right)^2\right] + (\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) + \frac{2}{3}\left(\frac{D_I}{\bar{R}_I}\right)^2\right], \quad \text{for } I = 2, 3, \ldots, N. \tag{A10c}$$

$$K_{I,I+1} = K_{I+1,I} = -(\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) - \frac{1}{3}\left(\frac{D_I}{\bar{R}_I}\right)^2\right], \quad \text{for } I = 1, 2, \ldots, N \tag{A10d}$$

$$\text{all other } K_{IJ} = 0. \tag{A10e}$$

With these specifications for M_{IJ}, K_{IJ}, b_I, the Galerkin solution of eqns (22) may be obtained by the same procedure described at the end of Section 4 except that t_{1I} ($I = 2, 3, \ldots, N$) and t_{2I} ($I = 1, 2, \ldots, N-1$) cannot be determined.

Alternatively, by focusing attention on the region of the Ith shell instead of the whole spherical region, we can obtain two equations to determine the contact stresses t_{1I} and t_{2I}. Specifically, multiplying eqn (A3) by $\xi^2\phi_I$ and $\xi^2\phi_{I+1}$, independently integrating over the region $[\xi_I, \xi_{I+1}]$, using eqns (A4), (A6) and the constitutive eqn (A2a) we obtain

$$\begin{aligned}\xi_I^2 t_{1I} = &-\rho_{0I}\bar{R}_I^2\left(\frac{1}{2}f_I - l_I\right)\\ &+ (\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) + \frac{2}{3}\frac{D_I^2}{\bar{R}_I^2}\right]U_I - 2\lambda\xi_I U_I\\ &- (\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) - \frac{1}{3}\frac{D_I^2}{\bar{R}_I^2}\right]U_{I+1}\\ &+ \rho_{0I}\bar{R}_I^2\left[\left(\frac{1}{4} - y_I^1 + y_I^2\right)\ddot{U}_I + \left(\frac{1}{4} - y_I^2\right)\ddot{U}_{I+1}\right]\end{aligned} \tag{A11a}$$

$$\begin{aligned}\xi_{I+1}^2 t_{2I} = &-\rho_{0I}\bar{R}_I^2\left(\frac{1}{2}f_I + l_I\right)\\ &- (\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) - \frac{1}{3}\frac{D_I^2}{\bar{R}_I^2}\right]U_I\\ &+ (\lambda + 2\mu)\left(\frac{\bar{R}_I^2}{D_I}\right)\left[\left(1 + \frac{1}{12}\frac{D_I^2}{\bar{R}_I^2}\right) + \frac{2}{3}\frac{D_I^2}{\bar{R}_I^2}\right]U_{I+1} + 2\lambda\xi_{I+1}U_{I+1}\\ &+ \rho_{0I}\bar{R}_I^2\left[\left(\frac{1}{4} - y_I^2\right)\ddot{U}_I + \left(\frac{1}{4} + y_I^1 + y_I^2\right)\ddot{U}_{I+1}\right]\end{aligned} \tag{A11b}$$

where we have used definitions (12f), (12g) and (30) with a, b, c defined by eqns (40). Finally, the equations of motion (22) may be obtained by using eqn (A11a) for $I = 1$, eqn (A11b) for $I = N$ and substituting eqn (A11) into the coupling eqn (13).